Proof Without Words: The Sum of Cubes—An Extension of Archimedes' Sum of Squares

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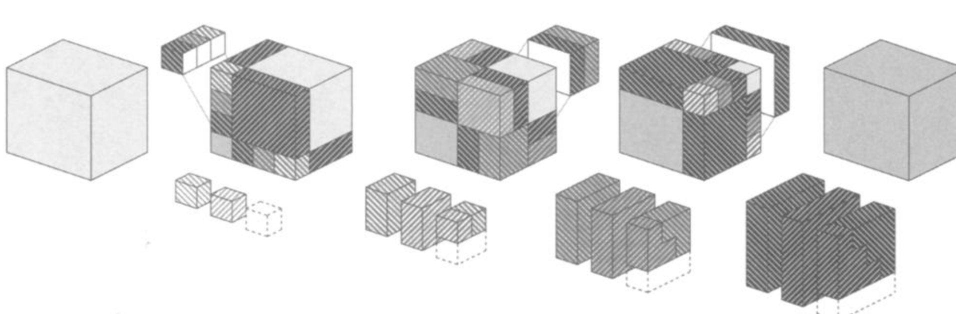
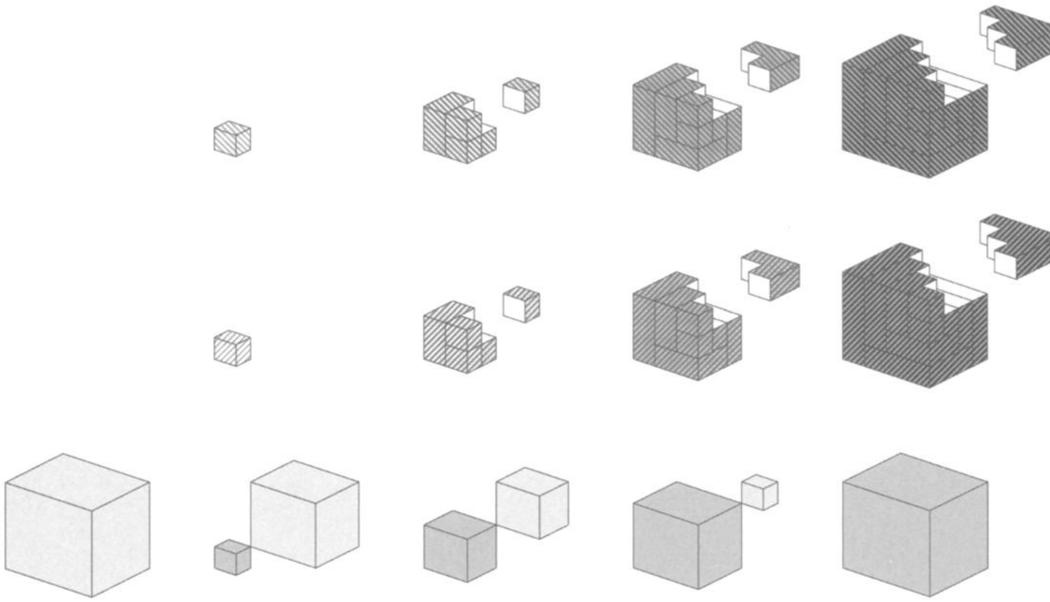
Archimedes' proposition determining a sum of squares has been displayed geometrically in a previous Proof Without Words [**1**]. We generalize to a sum of cubes, stating the result first as Archimedes might have, in the geometric language he used for his sum of squares. Then we give a proof befitting the geometric language.

> If a series of any number of lines be given, which exceed one another by an equal amount, and the difference be equal to the least, and if other lines be given equal in number to these and in quantity to the greatest, the cubes on the lines equal to the greatest, plus the cube on the greatest and the triplicate of the rectangular solids contained by the least and the squares of the lines exceeding one another by an equal amount, less the rectangular solids contained by the square on the least and those exceeding one another by an equal amount, will be the quadruple of all the cubes on the lines exceeding one another by an equal amount.

In modern symbolism,

$$(n+1)n^3 + 3\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i = 4\sum_{i=1}^{n} i^3.$$





Further investigation has revealed a generalization for arbitrary exponents. A geometric figure for a sum of fourth powers, which proves

$$(n+1)n^4 + 6\sum_{i=1}^{n} i^3 - 4\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = 5\sum_{i=1}^{n} i^4,$$

is challenging, but is underway via a diagram involving shadows of hypercubes.

Acknowledgment. Thank you to David Pengelley for suggestions and support, and to Roger Nelsen for the beautiful electronic layout of the figure.

REFERENCES

1. K. Kanim, Proof without words: How did Archimedes sum squares in the sand?, this MAGAZINE **74** (2001), 314–315.

An October Warning

Possibly Gilman ought not to have studied so hard. Non-Euclidean calculus and quantum physics are enough to stretch any brain; and when one mixes them with folklore, and tries to trace a strange background of multi-dimensional reality behind the ghoulish hints of the Gothic tales and the wild whispers of the chimney-corner, one can hardly expect to be wholly free from mental tension.... The professors at Miskatonic had urged him to slacken up.... But all these precautions came late in the day, so that Gilman had some terrible hints from the dreaded *Necronomicon*....

From *The Dreams in the Witch-House*, by H. P. Lovecraft